Rose's Linear Algebra

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Adapted from

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1 Vector Spaces

1.1 Vector Spaces

Linear algebra is the study of vector spaces and linear maps. Linear problems are easier to break into smaller problems than are nonlinear problems.

V is a **vector space** over a field F if V is an abelian group under **addition** $+$, V is an F-set under **scalar multiplication** $\cdot$, and $+, \cdot$ are distributive. The set $M_{m\times n}(F)$ of $m \times n$ matrices in F forms a vector space over F. The set $\mathcal{F}(D, V)$ of functions $f : D \to V$ forms a vector space over F such that

$$(f + g)(x) = f(x) + g(x)$$

$$(\lambda f)(x) = \lambda f(x)$$

$\emptyset \neq W \subseteq V$ is a **subspace** $W \leq V$ of V if W is a vector space under the same operations. Equivalently, W is closed:

$$\forall \vec{v}_1, \vec{v}_2 \in W(\lambda\vec{v}_1 + \vec{v}_2 \in W)$$

Such $W \subset V$ is a **proper subspace**. Such $\{\vec{0}\}$ is the **trivial subspace**.

$V \cap W$ is a subspace of V and of W. The **sum** is

$$V + W = \{\vec{v} + \vec{w} : \vec{v} \in V \wedge \vec{w} \in W\}$$

If $V \cap W = \{\vec{0}\}$, then this is the **direct sum** $V \oplus W$. If $V, W \leq U$, then $V \oplus W$ is the smallest subspace of U containing V, W, and $V \oplus W = U$ if and only if $\vec{u} = \vec{v} + \vec{w}$ is unique for all $\vec{u} \in U$.

A **linear combination** is

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

The **span** of $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is

$$\operatorname{Span} S = \{c_1\vec{v}_1 + \cdots + c_n\vec{v}_n : c_i \in F\} \leq V$$

with $\operatorname{Span}\emptyset = \{\vec{0}\}$. S is a **spanning set**, or **generating set**, for V if $\operatorname{Span} S = V$.

S is **linearly dependent** if there is a **linear dependence**

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$$

where some $c_i \neq 0$, and is otherwise **linearly independent**. An infinite set is linearly independent if and only if all finite subsets are linearly independent. If $S_1 \subseteq S_2$ and S_1 is linearly dependent, then S_2 is linearly dependent. If S is linearly independent and $\vec{v} \notin S$, then $S \cup \{\vec{v}\}$ is linearly independent if and only if $\vec{v} \notin \operatorname{Span} S$.

A **basis** is a linearly independent spanning set. If $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis and

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

then the **coordinate representation**, or **column vector**, of $\vec{v}$ with respect to $\mathcal{B}$ is

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

If $\mathcal{B} \neq \emptyset$ is finite, then $\mathcal{B}$ is a basis if and only if every $\vec{v}$ has a unique coordinate representation. Some standard bases are

$$\mathcal{E}_{\mathbb{F}^n} = \{\hat{e}_1, \ldots, \hat{e}_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{E}_{P_n(F)} = \{t^0, \ldots, t^n\}$$

V is **finite dimensional** if there is a finite spanning set S. The **existence theorem** states that, if $X \subseteq S$ is linearly independent, then there is a basis $X \subseteq \mathcal{B} \subseteq S$. The **exchange theorem**, or **replacement theorem**, states that there is an **exchange** $T \subseteq S$ such that $|T| = |X|$ and $\operatorname{Span}(X \cup (S \setminus T)) = V$. The **extension theorem** states that a linearly independent set is a subset of a basis. The **dimension** $\dim_F V$ is the cardinality of every basis. If $W \leq V$, then $\dim W \leq \dim V$. If $\dim W = \dim V$, then $W = V$.

We may account for infinite-dimensional spaces, for example Banach and Hilbert spaces, by defining infinite sums, as in the field of analysis. Alternatively, we may use Zorn's lemma: the maximal element of the set of linearly independent subsets under $\subseteq$ is a basis.

1.2 Linear Maps

A **linear map** is a homomorphism $T : V \to W$ preserving scalar multiplication, i.e. both spaces are over F. $\mathcal{L}(V, W) \leq \mathcal{F}(V, W)$ is the set of linear maps. A **linear isomorphism** is a bijective linear map.

The range, or **column space**, is $\mathcal{R}(T) \leq W$ and the **nullspace** is $\mathcal{N}(T) = \ker T \leq V$. The **rank** is $\operatorname{rank} T = \dim \mathcal{R}(T)$ and the **nullity** is $\operatorname{null} T = \dim \mathcal{N}(T)$. T uniquely maps $\mathcal{B}$ to a spanning set for $\mathcal{R}(T)$. The **rank–nullity theorem** states

$$\operatorname{rank} T + \operatorname{null} T = \dim V$$

T is injective if and only if $\operatorname{null} T = 0$. T is surjective only if $\operatorname{rank} T = \dim W$, and then if W is finite dimensional. If $\dim V = \dim W$ is finite, then

$$\operatorname{null} T = 0 \iff \operatorname{rank} T = \dim W$$

If T is a linear isomorphism, then $T(\mathcal{B}) = \mathcal{C}$ is a basis of W. V, W are isomorphic if and only if $\dim V = \dim W$.

The unique **matrix representation** of T with respect to $\mathcal{B},\mathcal{C}$ is

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{C}} & \cdots & [T(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix} \in M_{m\times n}(F)$$

satisfying

$$\forall \vec{v} \in V([T(\vec{v})]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}}[\vec{v}]_{\mathcal{B}})$$

and uniquely defines T. Every linear map, given bases, is equivalent to left multiplication by a unique matrix.

$$[UT]_{\mathcal{B}}^{\mathcal{D}} = [U]_{\mathcal{C}}^{\mathcal{D}}[T]_{\mathcal{B}}^{\mathcal{D}}$$

The **rank**, **nullity**, **column space**, and **nullspace** of a matrix A are those of its equivalent linear map L_A. Then

$$\mathcal{R}(A) = \operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$$

The **identity matrix** I_n has $(I_n)_{ij} = \delta_{ij}$, where δ is the **Kronecker delta**

$$\delta : (i,j) \mapsto \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$$

If $T \in \mathcal{L}(V)$, then

$$[T]_{\mathcal{B}} = I_n \iff T = I$$

T is a linear isomorphism if and only if $[T]_{\mathcal{B}}^{\mathcal{C}}$ is invertible. Then

$$[T^{-1}]_{\mathcal{C}}^{\mathcal{B}} = ([T]_{\mathcal{B}}^{\mathcal{C}})^{-1}$$

Every finite-dimensional vector space is isomorphic to F^n by

$$\phi_{\mathcal{B}} : V \to F^n : \vec{v} \mapsto [\vec{v}]_{\mathcal{B}}$$

The linear maps of finite-dimensional spaces are isomorphic to the matrices by

$$\Phi_{\mathcal{B}}^{\mathcal{C}} : \mathcal{L}(V,W) \to M_{m\times n}(F) : T \mapsto [T]_{\mathcal{B}}^{\mathcal{C}}$$

The **change of coordinates matrix** in V from $\mathcal{B}$ to $\mathcal{E}$ is the invertible matrix

$$Q_{\mathcal{B}}^{\mathcal{E}} = [I]_{\mathcal{B}}^{\mathcal{E}} = \begin{bmatrix} [\vec{v}_1]_{\mathcal{E}} & \cdots & [\vec{v}_n]_{\mathcal{E}} \end{bmatrix} \in M_n(F)$$

satisfying

$$\forall \vec{v} \in V([\vec{v}]_{\mathcal{E}} = (\phi_{\mathcal{E}} \circ \phi_{\mathcal{B}}^{-1})(\vec{v}) = Q_{\mathcal{B}}^{\mathcal{E}}[\vec{v}]_{\mathcal{B}})$$

For $T : V \to V$,

$$[T]_{\mathcal{B}} = Q_{\mathcal{E}}^{\mathcal{B}}[T]_{\mathcal{E}}Q_{\mathcal{B}}^{\mathcal{E}} = Q_{\mathcal{E}}^{\mathcal{B}}[T]_{\mathcal{E}}(Q_{\mathcal{E}}^{\mathcal{B}})^{-1}$$

For $T : V \to W$,

$$[T]^{\mathcal{C}}_{\mathcal{B}} = Q^{\mathcal{C}}_{\mathcal{F}} [T]^{\mathcal{F}}_{\mathcal{E}} Q^{\mathcal{E}}_{\mathcal{B}}$$

$\mathcal{E}, F$ are often the standard bases because $Q^{\mathcal{E}}_{\mathcal{B}}, [T]^{\mathcal{F}}_{\mathcal{E}}, Q^{\mathcal{F}}_{\mathcal{C}}$ are easy to compute.

$$f : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to F$$

is a **k-multilinear map** if it is a linear map when all but one entry are held constant. Then f is an **alternating k-form** on V if it is zero when two entries are equal. The set of alternating k-forms is the vector space $\bigwedge^k V^*$.

$$\dim V = n \implies \dim \bigwedge^n V^* = 1$$

A **T-invariant subspace** $W \leq V$ has $T(W) \leq W$. Then the **restriction** is

$$T_W : W \to W : \vec{w} \mapsto T(\vec{w})$$

A **T-cyclic subspace** is

$$\langle \vec{v} \rangle = \operatorname{Span}\{T^k(\vec{v}) : k \in \mathbb{N}\} \leq V$$

the smallest T-invariant subspace containing $\vec{v}$.

$$\dim |\vec{v}| = 1 \iff T(\vec{v}) = \lambda \vec{v}$$

$$\dim |\vec{v}| = k \iff |\vec{v}| = \operatorname{Span}\{\vec{v}, T(\vec{v}), \ldots, T^{k-1}(\vec{v})\}$$

The **dual space** is $V^* = \mathcal{L}(V, \mathbb{F})$.

1.3 Elementary Operations

The **elementary row operations** are the linear isomorphisms mapping the identity matrix to the **elementary matrices**:

- **Type I** $I_n \mapsto E_{ij}$: Swap rows i, j.
- **Type II** $I_n \mapsto E_i^{(\lambda)}$: Multiply row i by $\lambda \neq 0$.
- **Type III** $I_n \mapsto E_{ij}^{(\lambda)}$: Add row j scaled by λ to row i.

Matrices are **row equivalent** $A \sim B$ if a finite number of row operations makes the matrices equal. A row operation is equivalent to left multiplication by its elementary matrix. The respective inverses are $E_{ij}, E_i^{(\lambda^{-1})}, E_{ij}^{(-\lambda)}$. $\operatorname{rank} A$ is invariant under multiplication by invertible matrices, e.g. elementary matrices.

The **elementary column operations** are the elementary row operations on columns. A column operation is an isomorphism equivalent to right multiplication by its elementary matrix.

The **leading entry** of a row is the leftmost nonzero entry. A **row echelon form** is a matrix such that

- Each nonzero row is above each zero row.
- Each leading entry is to the right of that of the row above.
- Each entry below a leading entry is zero.

The **reduced row echelon form** is the unique echelon form such that

- Each leading entry is 1.
- Each leading entry is the only nonzero entry in the column.

A **pivot position** is the position of a leading entry in the reduced echelon form. A **pivot column** contains a pivot position. A **pivot** is a nonzero entry in a pivot position. **Row reduction** is the transformation of a matrix to an echelon form by row operations:

1. Select a nonzero entry in the leftmost column. Use a Type I row operation to move the entry to the top row.
2. Use Type III row operations to make all entries below the pivot zero.
3. Repeat steps 1–2, ignoring the top row and leftmost column, until the matrix is an echelon form.
4. Use Type II row operations to make each pivot 1.
5. Use Type III row operations to make all entries above the rightmost pivot zero.
6. Repeat step 5, ignoring the rightmost column, until the matrix is in the reduced echelon form.

Row operations preserve the **row space**, the span of the rows. Column operations preserve the column space. rank $A = r$ if and only if a finite number of row and column operations produces

$$D = \begin{bmatrix} I_r & 0_{r(n-r)} \\ 0_{(m-r)r} & 0_{(m-r)(n-r)} \end{bmatrix}$$

That is, there are elementary matrices such that

$$R_k \cdots R_1 A C_1 \cdots C_l = D$$

The pivot columns of the reduced echelon form form a basis for the column space. The nonzero rows of the reduced echelon form form a basis for the row space. The dimensions of the column space and the row space are equal.

Every invertible matrix is a product of elementary matrices. Then there is a sequence QP of row operations mapping A to I_n, and $A^{-1} = QPI_n$. That is,

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \sim \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

Then reduction of A to I_n gives A^{-1}.

$$\operatorname{rank} A = \operatorname{rank} A^T$$

If $S : U \to V$ and $T : V \to W$, then

$$\operatorname{rank} S - \operatorname{null} T \leq \operatorname{rank} TS \leq \min(\operatorname{rank} T, \operatorname{rank} S)$$

$$\max(\operatorname{null} S, \dim U - \operatorname{rank} T) \leq \operatorname{null} TS \leq \operatorname{null} T + \operatorname{null} S$$

1.4 Determinants

The **order-n determinant**, or **order-n oriented area**, of F is the unique $\det \in \bigwedge^n (F^n)^*$ such that

$$\det I_n = \begin{vmatrix} \quad I_n \quad \end{vmatrix} = 1$$

The determinant is the factor by which a linear map scales the hypervolume of a given shape. $(\vec{a}_1, \ldots, \vec{a}_n)$ is **positively oriented** if $\det A > 0$ and **negatively oriented** if $\det A < 0$.

The ijth **minor** of A is

$$\widetilde{A}_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & & & & \vdots \\ a_{(i-1)1} & & & & & a_{(i-1)n} \\ a_{(i+1)1} & & & & & a_{(i+1)n} \\ \vdots & & & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

The **minor determinant** is $\det \widetilde{A}_{ij}$. The **cofactor** is

$$C_{ij} = (-1)^{i+j} \det \widetilde{A}_{ij}$$

The **cofactor expansion** of $\det A$ along the ith row is

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \widetilde{A}_{ij}$$

Equivalently,

$$\begin{aligned} \det A = {} & (a_{11} a_{22} \cdots a_{(n-1)(n-1)} a_{nn}) + \cdots + (a_{1n} a_{21} \cdots a_{(n-1)(n-2)} a_{n(n-1)}) \\ & - (a_{1n} a_{2(n-1)} \cdots a_{(n-1)2} a_{n1}) - \cdots - (a_{11} a_{2n} \cdots a_{(n-1)3} a_{n2}) \end{aligned}$$

The parallelogram or parallelepiped determined by the columns of a matrix have hypervolume $|\det A|$.

$$A \in M_2(F) \implies \det A = a_{11} a_{22} - a_{12} a_{21}$$

$$A \in M_3(F) \implies \det A = \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)$$

$\det A = 0$ if and only if the columns or rows are linearly dependent. The row operations give

- Type I: $\det A \mapsto -\det A$.
- Type II: $\det A \mapsto \lambda \det A$.
- Type III: $\det A \mapsto \det A$.

$$\det AB = \det A \det B$$

$$\det A^T = \det A$$

A is invertible if and only if $\det A \neq 0$. Then

$$\det A^{-1} = \frac{1}{\det A}$$

Cramer's rule states that the unique solution to $A\vec{x} = \vec{b}$ has

$$x_k = \frac{1}{\det A} \det M_k$$

where

$$M_k = \begin{bmatrix} \vec{m}_1 & \cdots & \vec{m}_{k-1} & \vec{b} & \vec{m}_{k+1} & \cdots & \vec{m}_n \end{bmatrix}$$

The **adjugate**, or **classical adjoint**, is

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$$A \in M_2(F) \implies A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

1.5 Diagonalization

An **eigenvector**[1] $\vec{v} \neq \vec{0}$ with **eigenvalue** λ has

$$T(\vec{v}) = [T]_{\mathcal{E}} \vec{v} = \lambda \vec{v}$$

T is **diagonalizable** if there is a basis $\mathcal{H}$ of eigenvectors, an **eigenbasis**. Then $[T]_{\mathcal{H}}$ is diagonal. The **spectrum** is the set of eigenvalues.

[1]The terms "eigen" and "characteristic" are equivalent.

Eigenvalues are invariant across bases, and the coordinate representation preserves eigenvectors:

$$T(\vec{v}) = \lambda\vec{v} \iff [T]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} = \lambda[\vec{v}]_{\mathcal{B}}$$

The **characteristic polynomial** of T is

$$p(t) = \det([T]_{\mathcal{B}} - tI_n) = (-1)^n t^n + \cdots + c$$

The **characteristic equation** is

$$p(t) = 0$$

whose solutions are exactly the eigenvalues. The **eigenspace** $E_\lambda = \mathcal{N}([T]_{\mathcal{B}} - \lambda I_n)$ is exactly the set of eigenvectors and $\vec{0}$. Complex eigenvalues of real matrices always occur in conjugate pairs, and the corresponding eigenvectors are a conjugate pair.

Distinct eigenvalues give linearly independent eigenvectors. If $p(t)$ has $\dim V$ distinct roots, then T is diagonalizable. If T is diagonalizable, then $p(t)$ splits over F. The **geometric multiplicity** of λ is $\dim E_\lambda$.

$$\dim E_\lambda \leq \nu_\lambda$$

The following are equivalent:

- T is diagonalizable.
- $p(t)$ splits and $\dim E_{\lambda_i} = \nu_{\lambda_i}$ for all λ_i.
- $\sum_{i=1}^k \dim E_{\lambda_i} = n$.
- $V = \bigoplus_{i=1}^k E_{\lambda_i}$.

The characteristic polynomial $p_W(t)$ of a restriction T_W divides that of T. If $W = \langle\vec{w}\rangle$ and $\dim W = k$, then

$$\operatorname{Span}\{T^0(\vec{w}), \dots, T^{k-1}(\vec{w})\} = W$$

$$T^k(\vec{w}) + a_{k-1}T^{k-1}(\vec{w}) + \cdots + a_1 T(\vec{w}) + a_0\vec{w} = \vec{0}$$
$$\implies p_W(t) = (-1)^k(t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0)$$
$$p_W(T_W) = 0$$

The **Cayley–Hamilton theorem** states $p(T) = 0$. Then

$$T^{-1} = -\frac{1}{a_0}T^{k-1} - \frac{a_{k-1}}{a_0}T^{k-2} - \cdots - \frac{a_1}{a_0}I$$

2 Inner Product Spaces

2.1 Bilinear Forms

A **bilinear form** is a bilinear map.

$$[B]_{\mathcal{B}} = \begin{bmatrix} B(\vec{v}_1, \vec{v}_1) & \cdots & B(\vec{v}_1, \vec{v}_n) \\ \vdots & \ddots & \vdots \\ B(\vec{v}_n, \vec{v}_1) & \cdots & B(\vec{v}_n, \vec{v}_n) \end{bmatrix}$$

$$B(\vec{v}, \vec{w}) = [\vec{v}]_{\mathcal{B}}^T [B]_{\mathcal{B}} [\vec{w}]_{\mathcal{B}}$$

If V is finite dimensional, then $[B]_{\mathcal{C}} = (Q_{\mathcal{C}}^{\mathcal{B}})^T [B]_{\mathcal{B}} Q_{\mathcal{C}}^{\mathcal{B}}$, and B is symmetric if and only if $[B]_{\mathcal{B}}$ is symmetric. If B is symmetric, then its matrix is diagonalized by a product of type III elementary matrices.

If V is finite dimensional, then B is diagonalizable only if it is symmetric, and then if $\operatorname{char} F \neq 2$. **Sylvester's law of inertia** states that, if V is finite dimensional and B is symmetric, then the number of diagonal entries of each sign is independent of the diagonalizing basis. Then the **signature** (n_+, n_-, n_0) of is the number of diagonal entries of each sign. The **quadratic form** of a symmetric bilinear form is

$$K : V \to \mathbb{F} : \vec{v} \mapsto B(\vec{v}, \vec{v})$$

An **inner product space** $(V, \langle , \rangle)$ is V under the **inner product** $\langle , \rangle : V \times V \to F$, which is linear in the first entry

$$\langle \lambda \vec{v}_1 + \vec{v}_2, \vec{v}_3 \rangle = \lambda \langle \vec{v}_1, \vec{v}_3 \rangle + \langle \vec{v}_2, \vec{v}_3 \rangle$$

conjugate symmetric

$$\langle \vec{v}_2, \vec{v}_1 \rangle = \overline{\langle \vec{v}_1, \vec{v}_2 \rangle}$$

and **positive definite**

$$\vec{v} \neq 0 \implies \langle \vec{v}, \vec{v} \rangle > 0$$

The **norm**, or **length** or **magnitude**, is

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

A **unit vector** has $\|\vec{v}\| = 1$. $\vec{v}_1, \vec{v}_2$ are **perpendicular**, or **orthogonal**, if $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$, and **orthonormal** if they are also unit.

$$\langle \vec{0}, \vec{v} \rangle = 0$$

$$\|\vec{v}\| = 0 \iff \vec{v} = 0$$

$$\|\lambda \vec{v}\| = |\lambda| \|\vec{v}\|$$

$$\forall \vec{w} \langle \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle \implies \vec{v} = \vec{u}$$

The **Cauchy–Schwarz inequality** states

$$|\langle \vec{v}_1, \vec{v}_2 \rangle| \leq \|\vec{v}_1\| \|\vec{v}_2\|$$

with equality if and only if $\vec{v}_1, \vec{v}_2$ are parallel. The **triangle inequality** states

$$\|\vec{v}_1 + \vec{v}_2\| \leq \|\vec{v}_1\| + \|\vec{v}_2\|$$

with equality if and only if $\vec{v}_1, \vec{v}_2$ are parallel in the same direction. The **distance** between $\vec{v}_1$ and $\vec{v}_2$ is $\|\vec{v}_1 - \vec{v}_2\|$. The **Pythagorean theorem** states

$$\langle \vec{u}, \vec{v} \rangle = 0 \iff \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

The **angle** θ between $\vec{v}_1, \vec{v}_2$ has

$$\cos\theta = \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\|\vec{v}_1\| \|\vec{v}_2\|}$$

The inner product is **sesquilinear**[2], that is, linear in the first entry and **conjugate linear**, or **antilinear**, in the second:

$$\langle \vec{v}_3, \lambda\vec{v}_1 + \vec{v}_2 \rangle = \overline{\lambda}\langle \vec{v}_3, \vec{v}_1 \rangle + \langle \vec{v}_3, \vec{v}_2 \rangle$$

If $F = \mathbb{R}, \mathbb{C}$, then there is exactly one inner product on V such that $\mathcal{B}$ is orthonormal.

2.2 Orthogonality

The **orthogonal complement** of $U \leq V$ is

$$U^\perp = \{\vec{v} \in V : \forall \vec{u} \in U(\langle \vec{v}, \vec{u} \rangle = 0\} \leq V$$

Then $U \cap U^\perp = \{\vec{0}\}$ and $U \leq (U^\perp)^\perp$. If $U = \operatorname{Span}\mathcal{B}$ and $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ is orthogonal, then $\mathcal{B}$ is a basis for U:

$$\vec{u} = \sum_{j=1}^{n} \frac{\langle \vec{u}, \vec{u}_j \rangle}{\|\vec{u}_j\|^2} \vec{u}_j$$

Also, $V = U \oplus U^\perp$:

$$\vec{v} = \vec{u} + \vec{w} = \sum_{j=1}^{n} \frac{\langle \vec{v}, \vec{u}_j \rangle}{\|\vec{u}_j\|^2} \vec{u}_j + (\vec{v} - \vec{u})$$

The **Gram–Schmidt process** states that, if S is linearly independent, then there is an orthogonal basis of $\operatorname{Span} S$ with $\vec{u}_1 = a_1\vec{s}_1$ and

$$\vec{u}_k = a_k \left(\vec{s}_k - \sum_{j=1}^{k-1} \frac{\langle \vec{s}_k, \vec{u}_j \rangle}{\|\vec{u}_j\|^2} \vec{u}_j \right)$$

[2] In physics convention, the inner product is written $\langle \vec{v}_2 \mid \vec{v}_1 \rangle$, so the first entry is conjugate linear and the second entry is linear.

Every countable-dimensional inner product space has an orthonormal basis. If U is infinite dimensional, then the **Fourier series** of $\vec{v}$ is

$$\sum \langle \vec{v}, \vec{u}_n \rangle \vec{u}_n$$

and $\langle \vec{x}, \vec{u}_n \rangle$ are the **Fourier coefficients**.

The **adjoint** of $T \in \mathcal{L}(V, W)$ is

$$T^* : W \to V : \langle T^*(\vec{w}), \vec{v} \rangle = \langle \vec{w}, T(\vec{v}) \rangle$$

If T^* exists, then it is unique and linear.

$$(T^*)^* = T, (TS)^* = S^*T^*, (\lambda T + S)^* = \bar{\lambda} T^* + S^*$$

If V, W have finite bases $\mathcal{B}, \mathcal{C}$, respectively, then T^* exists and

$$[T^*]_{\mathcal{C}}^{\mathcal{B}} = ([T]_{\mathcal{B}}^{\mathcal{C}})^*$$

The **fundamental subspaces theorem** states

$$\mathcal{R}(T^*)^{\perp} = \mathcal{N}(T)$$

The **Riesz representation theorem** states that, if V is finite dimensional and $g \in V^*$, then there is a unique $\vec{y} \in V$ such that $g(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$. Then

$$\phi : V \to V^* : \vec{y} \mapsto g_{\vec{y}}$$

is an isomorphism. Every linear map with finite-dimensional domain has an adjoint.

T is **normal** if $TT^* = T^*T$. Then

- $\|T(\vec{v})\| = \|T^*(\vec{v})\|$,
- $T - tI$ is normal with adjoint $T^* - \bar{t}I$,
- $T(\vec{v}) = \lambda \vec{v} \iff T^*(\vec{v}) = \bar{\lambda}\vec{v}$, and
- the eigenvalues have orthogonal eigenvectors

T is **self adjoint**, or **Hermitian**, if $T^* = T$. Then

- if $W \leq V$ is T-invariant, then T_W is self adjoint,
- if V is finite dimensional, then T has an eigenvalue, and
- the eigenvalues are real

T is **skew Hermitian** if $T^* = -T$, and is then normal and not self-adjoint.

Schur's lemma states that, if $p(t)$ splits, then there is an orthonormal basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is upper triangular.

A **linear isometry** has $\|T(\vec{v})\| = \|\vec{v}\|$. A **unitary operator** has $T^*T = I = TT^*$. If $\mathbb{F} = \mathbb{R}$, then this is an **orthogonal operator**. T is a unitary operator only if it is a linear isometry, and then if V is finite dimensional. If V is finite dimensional, then the following are equivalent:

- T is a unitary operator.
- T preserves the inner product.
- T has an orthonormal basis.
- T maps every orthonormal basis to an orthonormal basis.

If $A \in M_n(\mathbb{R})$, then A is orthogonal if and only if its columns are orthonormal in $\mathbb{R}^n$. If $A \in M_n(\mathbb{C})$, then A is unitary if and only if its columns are orthonormal in $\mathbb{C}^n$.

A and B are **unitarily equivalent** if there is a unitary matrix U such that $B = U^*AU$. $A \in M_n(\mathbb{C})$ is normal if and only if it is unitarily equivalent to a diagonal matrix. $A \in M_n(\mathbb{R})$ is self adjoint if and only if it is orthogonally equivalent to a diagonal matrix.

2.3 Projections

A **projection** has $V = \mathcal{R}(T) \oplus \mathcal{N}(T)$ and $T_{\mathcal{R}(T)} = I_{\mathcal{R}(T)}$. Equivalently, $T(\vec{r} + \vec{n}) = \vec{r}$. A **projection matrix** A has L_A a projection. T is a projection if and only if $T^2 = T$.

An **orthogonal projection** is a projection with $\mathcal{N}(T) = \mathcal{R}(T)^\perp$ and $\mathcal{R}(T) = \mathcal{N}(T)^\perp$. Equivalently, if $V = W \oplus W^\perp$, then the orthogonal projection π_W has $\mathcal{R}(\pi_W) = W$ and $\mathcal{N}(\pi_W) = W^\perp$. The **complementary orthogonal projection** $\pi_W^\perp = I - \pi_W$ has $\mathcal{R}(\pi_W^\perp) = W^\perp$ and $\mathcal{N}(\pi_W^\perp) = W$. If T is a projection, then T is an orthogonal projection if and only if it is self adjoint.

The **spectral theorem** states that, if V is finite dimensional over $\mathbb{C}$ ($\mathbb{R}$), then T is normal (self adjoint) if and only if T has an orthonormal eigenbasis. Then

- $E_j^\perp = \bigoplus_{i \neq j} E_i$,
- $\pi_{E_i}\pi_{E_j} = 0$ for $i \neq j$,
- the **resolution of the identity** is $I = \sum \pi_{E_i}$, and
- the **spectral decomposition** is $T = \sum \lambda_i \pi_{E_i}$.

π_W minimizes the distance between $\vec{v}$ and W:

$$\|\vec{v} - \pi_W(\vec{v})\| \leq \|\vec{v} - \vec{w}\|$$

If V and W are finite dimensional and $r = \operatorname{rank} T$, then there are respective orthonormal bases $\mathcal{B}$ and $\mathcal{C}$ and unique **singular values**

$$\sigma_1 \geq \cdots \geq \sigma_r \geq 0 = \sigma_{r+1} = \cdots = \sigma_{\min\{m,n\}}$$

such that

$$T(\vec{v}_j) = \begin{cases} \sigma_j \vec{w}_j & ,j \leq r \\ \vec{0} & ,j > r \end{cases}, T^*(\vec{w}_j) = \begin{cases} \sigma_j \vec{v}_j & ,j \leq r \\ \vec{0} & ,j > r \end{cases}, T^*T(\vec{v}_j) = \begin{cases} \sigma_j^2 \vec{v}_j & ,j \leq r \\ \vec{0} & ,j > r \end{cases}$$

Further,

$$[T]_{\mathcal{E}} = P[T]_{\mathcal{B}}^{\mathcal{C}}Q^* = \begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_m \end{bmatrix} \begin{bmatrix} \mathrm{diag}(\sigma_1, \dots, \sigma_r) & O \\ O & O \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}^*$$

The **singular value decomposition** has

$$T(\vec{v}) = \sum \sigma_j \vec{w}_j \vec{v}_j^* \vec{v}$$

The **pseudoinverse** has

$$T^\dagger(\vec{w}_j) = \begin{cases} \frac{1}{\sigma}\vec{v}_j & , j \leq r \\ \vec{0} & , j > r \end{cases}$$

$$T^\dagger T = \pi_{\mathcal{N}(T)}^\perp, TT^\dagger = \pi_{\mathcal{R}(T)}$$

The **method of least squares** states that, if $\{(t_i, y_i) : 1 \leq i \leq m\}$ is a dataset with best-fitting line $y = c_0 t^0 + \cdots + c_{n-1}t^{n-1}$, then y minimizes

$$\sum_{i=1}^m |y_i - c_0 t_i^0 - \cdots - c_{n-1}t_i^{n-1}|^2 = \|\vec{y} - A\vec{c}\|^2$$

If there are at least n distinct data points, then

$$\vec{c} = A^\dagger \vec{y} = (A^T A)^{-1} A^T \vec{y}$$

2.4 Canonical Forms

A **Jordan block** is

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \lambda \end{bmatrix}$$

or any 1×1 matrix. A **Jordan canonical form** is $\mathrm{diag}(J_1, \dots, J_m)$. A **Jordan canonical basis** makes $[T]_\beta$ a Jordan canonical form.

The **generalized eigenspace** is

$$K_\lambda = \{\vec{v} \in V : \exists k \in \mathbb{N}((T - \lambda I)^k(\vec{v}) = \vec{0})\} = \bigcup_{k \in \mathbb{N}} \mathcal{N}(T - \lambda I)^k$$

If λ is an eigenvalue, then

- $E_\lambda \leq K_\lambda \leq V$,
- K_λ is T invariant, and

- if K_λ is finite dimensional and $\mu \neq \lambda$, then
  - $(T-\mu I)_{K_\lambda}$ is an isomorphism, and
  - if μ is an eigenvalue, then $K_\lambda \cap K_\mu = \{\vec{0}\}$.

If $p(t)$ splits, then

- $K_{\lambda_i} = \mathcal{N}(T-\lambda_i I)^{\nu_i}$,
- $\dim K_{\lambda_i} = \nu_i$, and
- $V = \bigoplus K_{\lambda_i}$.

A **cycle** is

$$\mathcal{B}_{\vec{v}} = \{(T-\lambda I)^{k-1}(\vec{v}), \dots, (T-\lambda I)^0(\vec{v})\}$$

where $\vec{v} \neq \vec{0}$ and k is minimal such that $(T-\lambda I)^k(\vec{v}) = \vec{0}$. Then $\mathcal{B}_{\vec{v}}$ is linearly independent and $[T_{\operatorname{Span}\mathcal{B}_{\vec{v}}}]_{\mathcal{B}_{\vec{v}}}$ is a Jordan block with λ. If K_λ is finite dimensional, then it has a basis

$$\mathcal{B}_\lambda = \bigcup \mathcal{B}_{\vec{v}_i}$$

where $\mathcal{B}_{\vec{v}_i} \cap \mathcal{B}_{\vec{v}_j} = \{\vec{0}\}$ for $i \neq j$. If $p(t)$ splits, then $\bigcup \mathcal{B}_{\lambda_i}$ is a Jordan canonical basis.

The unique **dot diagram** of K_λ with a Jordan canonical basis from $\beta_{\vec{v}_1}, \dots, \beta_{\vec{v}_n}$ of nonincreasing size has n columns of $|\beta_{\vec{v}_i}|$ dots each going downward, representing the respective elements of $\beta_{\vec{v}_i}$ going rightward. The vectors associated with the first r rows form a basis of $\mathcal{N}(T-\lambda I)^r$. If r_i is the number of dots in row i, then

$$r_1 = (T-\lambda I) = \dim V - \operatorname{rank}(T-\lambda I)$$

$$r_i = (T-\lambda I)^i - (T-\lambda I)^{i-1} = \operatorname{rank}(T-\lambda I)^{i-1} - \operatorname{rank}(T-\lambda I)^i$$

The Jordan canonical form is unique up to reordering.

The **companion matrix** of a monic polynomial of degree k is

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -a_{k-1} \end{bmatrix}$$

If

$$p(t) = (-1)^n(\phi_1(t))^{\nu_1}\cdots(\phi_k(t))^{\nu_k}$$

where $\phi_i(t)$ is an irreducible monic polynomial, then a **rational canonical basis** gives the unique **rational canonical form**

$$[T]_\beta = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & C_r \end{bmatrix}$$

where C_i is the companion matrix of $(\phi_i(t))^{s_i}$ with $s_i \leq \nu_j$. The rational canonical form always exists.

A Matrices

The set of $m \times n$ matrices is $M_{mn}(F)$. A **diagonal entry** has equal row and column indices. The **main diagonal** is the diagonal entries. A **diagonal matrix** is

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

The **zero matrix** 0 has all entries zero.

Two matrices are **equal** if they have the same entries. **Addition** $A+B$ and **scalar multiplication** kA are entry-wise. The **row–column rule** states that, if $A \in M_{mn}(F), B \in M_{no}(F)$, then **matrix multiplication** is defined by

$$AB = \begin{bmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & \cdots & a_{11}b_{1o} + \cdots + a_{1n}b_{no} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & \cdots & a_{m1}b_{1o} + \cdots + a_{mn}b_{no} \end{bmatrix}$$

Matrix multiplication is associative, distributive, and scalar associative.

$$I_m A = A = A I_n$$

The **transpose** is

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$

The transpose is additively distributive and scalar associative.

$$(A^T)^T = A$$

$$(AB)^T = B^T A^T$$

A **symmetric matrix** has $A^T = A$. The **scalar product**, or **inner product** or **dot product**, is

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^T \vec{v}_2$$

The **outer product** is

$$\vec{v}_1 \otimes \vec{v}_2 = \vec{v}_1 \vec{v}_2^T$$

A **square matrix** $A \in M_{nn}(F) = M_n(F)$ is **invertible**, or **non-singular**, if it has an inverse, and is otherwise **singular**.

$$(A^T)^{-1} = (A^{-1})^T$$

A **partition** is a submatrix. In addition and scalar multiplication, a partition can be treated as a single entry. A, B are **conformable** if the columns of

A are partitioned in the same way as the rows of B. In **block multiplication** of conformable matrices, a partition can be treated as a single entry. If A is partitioned by column and B is partitioned by row, then AB is a sum of outer products. A **block upper triangular matrix** has only zero blocks to the left of the diagonal. A **block diagonal matrix** has only zero blocks off the main diagonal. A^{-1} can be found by a matrix-wise system of equations, e.g. finding the inverses of the diagonal blocks.

A and B are **similar** if $A = CBC^{-1}$ for some C. Then B is **conjugated** by C.

A **unit lower triangular matrix** is a matrix with only zeros to the right of the main diagonal and only ones on the diagonal. The **LU factorization** of a matrix which can be row reduced without row interchanges is the product of a unit lower triangular matrix and a reduced echelon form of the matrix. The matrix equation

$$A\vec{x} = LU\vec{x} = \vec{b}$$

can be solved by using the solution of the matrix equation

$$L\vec{y} = \vec{b}$$

to solve the matrix equation

$$U\vec{x} = \vec{y}$$

A unit lower triangular matrix can be constructed as follows:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.

2. Place entries in L such that the same sequence of row operations reduces L to I.

It follows that the number of columns in the unit lower triangular matrix is the same as the number of rows in the original matrix.

A **permutation** of a matrix is a row equivalent matrix by row interchanges. A **permuted lower triangular matrix** is a matrix which can be permuted into a lower triangular matrix. If row interchanges are necessary to reduce the original matrix to an echelon form, then the construction of a unit permuted lower triangular matrix gives a **permuted LU factorization** which can be solved similarly to an LU factorization by reducing the unit permuted lower triangular matrix from left to right.

The **QR factorization** of a matrix with linearly independent columns is the product of a matrix whose columns form an orthonormal basis for the column space of the original matrix and an upper triangular invertible matrix with positive entries on its diagonal.

B Linear Systems

A **linear equation** is

$$a_1x_1 + \cdots + a_nx_n = b$$

A **linear system** is

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

A **solution** is a value assignment that solves every equation in the system. The **solution set** is the set of solutions. Two systems are **equivalent** if the solution sets are equal. A system is **consistent** if there is a solution, and is otherwise **inconsistent**. The **coefficient matrix** is

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

The **augmented coefficient matrix** is

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \ddots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Since row operations preserve the row space, row-equivalent linear systems have the same solution set. Then row operations may be used to solve linear systems as augmented coefficient matrices. A **basic variable** corresponds to a pivot column, and a **free variable** does not. A **parametric solution set** is parameterized by the free variables. The system is consistent if and only if the augmented column is not a pivot column. Then there are infinitely many solutions if there is a free variable, and otherwise a unique solution. **Back-substitution** is the solution of a linear system by row reduction:

1. Find an echelon form of the augmented matrix. If the augmented column is a pivot column, then the system is inconsistent.

2. Find the reduced echelon form.

3. Solve each corresponding linear equation for the basic variable. This gives a parametric solution set.

The **matrix equation**

$$A\vec{x} = \vec{b}$$

has a solution if and only if $\vec{b} \in \mathcal{R}(A)$. The **row theorem** states that $\mathcal{R}(A) = F^m$ if and only if A has a pivot position in every row. The **homogeneous equation**

$$A\vec{x} = \vec{0}$$

has a nontrivial solution if and only if there is a free variable.

The solution set of a homogeneous equation can be represented as a **parametric vector equation**

$$\vec{x} = t_1\vec{v}_1 + \cdots + t_n\vec{v}_n$$

equivalent to

$$\mathcal{N}(A) = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$$

with $\text{null}\, A$ free variables. The solution set of a nonhomogeneous equation is a line

$$\vec{x} = \vec{p} + t\vec{v}_h$$

where $\vec{v}_h$ is a homogeneous solution and $\vec{p}$ is a nonhomogeneous solution.

The **column theorem** states that the homogeneous equation has only the trivial solution if and only if A has a pivot in every column. T is injective if and only if the homogeneous equation has only the trivial solution.

C Special Linear Transformations

A **shear map** is a linear map with $V = W$.

$$T : V \to V : \vec{v} \mapsto r\vec{v}$$

is a **contraction** if $0 \leq r \leq 1$ and a **dilation** if $r > 1$.

- Reflections
  - Reflection through the x_1-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
  - Reflection through the x_2-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
  - Reflection through the line $x_2 = x_1$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
  - Reflection through the line $x_2 = -x_1$: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
  - Reflection through the origin: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
- Contractions and Expansions
  - Horizontal contraction and expansion: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
  - Vertical contraction and expansion: $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
- Shears

- Horizontal shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
- Vertical shear: $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

- Projections
  - Projection onto the x_1-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
  - Projection onto the x_2-axis: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

D Special Inner Product Spaces

If $F = \mathbb{R}$, then the inner product is bilinear and symmetric. The **Euclidean space** is $\mathbb{R}^n$ under the **dot product**, or **scalar product**,

$$\cdot : (\vec{v}_1, \vec{v}_2) \mapsto \vec{v}_2^T \vec{v}_1 = \sum_{i=1}^{n} v_{1i} v_{2i}$$

A **weighted inner product** is

$$\langle , \rangle : (\vec{v}_1, \vec{v}_2) \mapsto \sum_{i=1}^{n} a_i v_{1i} v_{2i}$$

A symmetric **positive-definite matrix** $A \in M_2(\mathbb{R})$ has $\vec{v}^T A \vec{v} > 0$ for all $\vec{v} \neq \vec{0}$. All inner products on $\mathbb{R}^n$ are under such

$$\langle , \rangle : (\vec{v}_1, \vec{v}_2) \mapsto \vec{v}_2^T A \vec{v}_1$$

The **Hermitian space** is $\mathbb{C}$ under the **Hermitian product**

$$\langle , \rangle : (\vec{v}_1, \vec{v}_2) \mapsto \vec{v}_2^* \vec{v}_1 = \sum_{i=1}^{n} \vec{v}_{1i} \overline{\vec{v}}_{2i}$$

where the **conjugate transpose** is

$$A^* = \overline{A}^T = \begin{bmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{bmatrix}$$

A **weighted inner product** is

$$\langle , \rangle : (\vec{v}_1, \vec{v}_2) \mapsto \sum_{i=1}^{n} a_i \vec{v}_{1i} \overline{\vec{v}}_{2i}$$

with real weights. A **Hermitian matrix**, or **self-adjoint matrix**, $A \in M_n(\mathbb{C})$ has $A^* = A$. A **positive-definite matrix** has also $\vec{v}^* A \vec{v} > 0$ for all $\vec{v} \neq \vec{0}$.

If $F = \mathbb{R}, \mathbb{C}$ and $A, B \in M_{mn}(F)$, then the **Frobenius inner product** is

$$\langle , \rangle : (A, B) \mapsto \mathrm{tr}(B^* A)$$

where the **trace** is

$$\mathrm{tr}\, A = \sum_{i=1}^{n} a_{ii}$$

If $[a, b] \subseteq \mathbb{R}$ and F is the continuous functions $[a, b] \to F$, then the L^2 **inner product** is

$$\langle , \rangle : (f, g) \mapsto \int_a^b f(t)\overline{g(t)}\mathrm{d}t$$

A **square-summable sequence** (s_n) has

$$\sum_{n=1}^{\infty} |s_n|^2 < \infty$$

Then the ℓ^2 **inner product** is

$$\langle , \rangle : ((s_n), (t_n)) \mapsto \sum_{n=1}^{\infty} s_n \overline{t_n}$$